Capstone Design

Engineering Economics III

Interest Formulas

• Uniform annual series and future value
• Uniform annual series and present value
• Gradient series
• Effective interest rate
Uniform annual series and future value

• Suppose that there is a series of $n$ uniform payments, uniform in amount and uniformly spaced such as a payment every year.
• Let $A$ be the amount of each uniform payment.
• Let $F$ be a future, single amount equivalent to the series, with $F$ occurring at the same time as the last $A$ payment.
• Then the relationship between $F$ and $A$ is:

$$F = A \left[ (1 + i)^n - 1 \right] / i$$

Uniform annual series and future value (example)

• Example: If $100 is invested at the end of each year for the next ten years in a savings account that pays 5% interest, how much will be in the account immediately after the tenth payment?
  - $F$ is the unknown
  - $A = $100 per year
  - $i = 5\%$, understood to be 5% per year, compounded annually
  - $n = 10$ years
  - $F = A \left[ (1 + 0.05)^{10} - 1 \right] / 0.05$
    - $= $100 \left[ (1.05)^{10} - 1 \right] / 0.05$
    - $= $100 (0.6289 / 0.05) = $1,258.00
• Interest rate tables can also be used to calculate the value
Some basic variable definitions

- \( P \) = a present sum of money
- \( F \) = a future sum of money
  - The future sum \( F \) is an amount, \( n \) interest periods from the present, that is equivalent to \( P \) with interest rate \( i \)
- \( i \) = interest rate per interest period (stated as a decimal value)
- \( n \) = number of interest periods
- \( A \) = an end-of period cash receipt or disbursement in a uniform series, continuing for \( n \) periods, the entire series is equivalent to \( P \) or \( F \) at interest rate \( i \)
- \( G \) = the gradient (change) in cash flow (+/-) from period to period

Interest Rate Tables

- Lookup tables with pre-calculated values assist in determining values for \( A \), \( F \), or \( P \) given \( A \), \( F \), \( P \), and \( G \) (if appropriate)
- Generally formulated as a triple

\[ F = A \left( \frac{F}{A}, 5\%, 10 \right) \]

- Desired value
- \( F \)
- \( A \)
- \( \left( \frac{F}{A}, 5\%, 10 \right) \)
- Interest periods
- Given value
- Value from table
- Interest rate
Interest Rate Table (example)

- From previous example
- $A=100.00$
- $i=5\%$
- $n=10$
- Given $A$, lookup $F/A$ value from table to calculate $F$
  - $F/A = 12.578$

$$F = A \cdot (F/A,5\%,10) = 100 \cdot (12.578) = 1,258.00$$

Uniform annual series and present value

- Suppose that there is a series of $n$ uniform payments, uniform in amount and uniformly spaced such as a payment every year.
- Let $A$ be the amount of each uniform payment.
- Let $P$ be a single amount equivalent to the series, with $P$ occurring one period before the first $A$ payment.
  - Note that although $P$ is an abbreviation of "Present," the single amount $P$ may actually occur in the future as long as it occurs exactly one period before the first $A$ payment.
- The relationship between $P$ and $A$ is:

$$P = A \left[ \frac{(1 + i)^n - 1}{i} \right] \left[ \frac{1}{(1 + i)^n} \right]$$
Uniform annual series and present value (example)

• Example: Suppose that a recent college graduate has $3,000 available as a down payment on a new car.
• The graduate can afford a uniform car loan payment of no more than $500 per month for 48 months, beginning one month from now.
• Interest is 6% compounded monthly.
• What is most that the graduate can spend today on a new car?

Uniform annual series and present value (example – continued)

• Let \( X \) = most can spend (budget).
• \( X = P + $3,000 \)
• \( A = $500 \) per month
• \( i = 0.5\% \) per month
• \( n = 48 \) months

\[
P = A \left[ \frac{(1 + i)^n - 1}{i (1 + i)^n} \right] = \frac{500 \left[ (1.005)^{48} - 1 \right]}{0.005 (1.005)^{48}} = $21,290
\]

• Or, using the 0.5\% interest table, which is quicker:

\[
P = A(P/A, 0.5\%, 48) = $500 (42.580) = $21,290
\]

\[
X = $21,290 + $3,000 = $24,290
\]
Gradient Series

• Suppose that there is a series of \( n \) payments uniformly spaced, but differing from one period to the next by a constant.
• The change or \textit{gradient} from one period to the next is denoted \( G \).
• Let \( A_1 \) be the payment at EOY 1.
  
  – EOY = End of year
  – NCF = net cash flow

Gradient Series (examples)

• Example 1:

\[
\begin{array}{c|c}
\text{EOY} & \text{NCF ($)} \\
\hline
1 & 100 \\
2 & 150 \\
3 & 200 \\
4 & 250 \\
\end{array}
\]

– \( A_1 = $100 \)
– \( G = + $50 \)

• Example 2:

\[
\begin{array}{c|c}
\text{EOY} & \text{NCF ($)} \\
\hline
1 & 100 \\
2 & 90 \\
3 & 80 \\
4 & 70 \\
\end{array}
\]

– \( A_1 = $100 \)
– \( G = - $10 \)
Gradient Series (examples)

- To find the present worth, at EOY 0, of a gradient series that begins EOY 1, use:
  \[ P = A_1 \left( \frac{P}{A,i\%,n} \right) + G \left( \frac{P}{G,i\%,n} \right) \]

- To find the annual equivalent (A series) of a gradient series that begins EOY 1, use:
  \[ A = A_1 + G \left( \frac{A}{G,i\%,n} \right) \]

- Example:
  - Find the present value at 10% interest of the series of payments given in Example 2 above.
    \[ P = 100 \left( \frac{P}{A,10\%,4} \right) + (-10) \left( \frac{P}{G,10\%,4} \right) \]
    \[ = 100 (3.170) - 10 (4.378) = 273 \]

Effective Interest Rate

- An interest rate takes two forms:
  - nominal interest rate and effective interest rate.
- The nominal interest rate does not take into account the compounding period.
- The effective interest rate does take the compounding period into account and thus is a more accurate measure of interest charges.
- A statement that the "interest rate is 10%" means that interest is 10% per year, compounded annually.
- In this case, the nominal annual interest rate is 10% and the effective annual interest rate is also 10%.
- However, if compounding is more frequent than once per year, then the effective interest rate will be greater than 10%.
- The more often compounding occurs, the higher the effective interest rate.
Effective Interest Rate (continued)

- The relationship between nominal annual and effective annual interest rates is:
  \[ i_a = \left[ 1 + \left( \frac{r}{m} \right) \right]^m - 1 \]
- where \( i_a \) is the effective annual interest rate
- \( r \) is the nominal annual interest rate,
- \( m \) is the number of compounding periods per year

Effective Interest Rate (example)

- Example: A credit card company charges 21\% interest per year, compounded monthly. What effective annual interest rate does the company charge?
  
  \[ r = 0.21 \text{ per year} \]
  \[ m = 12 \text{ months per year} \]
  \[ i_a = \left[ 1 + \left( \frac{0.21}{12} \right) \right]^{12} - 1 \]
  \[ = \left[ 1 + 0.0175 \right]^{12} - 1 \]
  \[ = (1.0175)^{12} - 1 = 1.2314 - 1 \]
  \[ = 0.2314 = 23.14\% \]