ECE380 Digital Logic

Introduction to Logic Circuits:
Boolean algebra

Axioms of Boolean algebra

- Boolean algebra: based on a set of rules derived from a small number of basic assumptions (axioms)

1a 0·0=0
1b 1+1=1
2a 1·1=1
2b 0+0=0
3a 0·1=1·0=0
3b 1+0=0+1=1
4a If x=0 then x′=1
4b If x=1 then x′=0
**Single-Variable theorems**

- From the axioms are derived some rules for dealing with single variables
  - 5a \( x \cdot 0 = 0 \)
  - 5b \( x + 1 = 1 \)
  - 6a \( x \cdot 1 = x \)
  - 6b \( x + 0 = x \)
  - 7a \( x \cdot x = x \)
  - 7b \( x + x = x \)
  - 8a \( x \cdot x' = 0 \)
  - 8b \( x + x' = 1 \)
  - 9 \( x'' = x \)

- Single-variable theorems can be proven by perfect induction

- Substitute the values \( x = 0 \) and \( x = 1 \) into the expressions and verify using the basic axioms

**Duality**

- Axioms and single-variable theorems are expressed in pairs
  - Reflects the importance of duality
- Given any logic expression, its dual is formed by replacing all + with \( \cdot \), and vice versa and replacing all 0s with 1s and vice versa
  - \( f(a, b) = a + b \)  dual of \( f(a, b) = a \cdot b \)
  - \( f(x) = x + 0 \)  dual of \( f(x) = x \cdot 1 \)

- The dual of any true statement is also true
## Two & three variable properties

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>10a.</td>
<td>$x \cdot y = y \cdot x$</td>
<td><strong>Commutative</strong></td>
</tr>
<tr>
<td>10b.</td>
<td>$x + y = y + x$</td>
<td></td>
</tr>
<tr>
<td>11a.</td>
<td>$x \cdot (y \cdot z) = (x \cdot y) \cdot z$</td>
<td><strong>Associative</strong></td>
</tr>
<tr>
<td>11b.</td>
<td>$x + (y + z) = (x + y) + z$</td>
<td></td>
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<tr>
<td>12a.</td>
<td>$x \cdot (y + z) = x \cdot y + x \cdot z$</td>
<td><strong>Distributive</strong></td>
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<tr>
<td>12b.</td>
<td>$x + y \cdot z = (x + y) \cdot (x + z)$</td>
<td></td>
</tr>
<tr>
<td>13a.</td>
<td>$x + x \cdot y = x$</td>
<td><strong>Absorption</strong></td>
</tr>
<tr>
<td>13b.</td>
<td>$x \cdot (x + y) = x$</td>
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</tr>
<tr>
<td>14a.</td>
<td>$x \cdot y + x \cdot y' = x$</td>
<td><strong>Combining</strong></td>
</tr>
<tr>
<td>14b.</td>
<td>$(x + y) \cdot (x + y') = x$</td>
<td></td>
</tr>
<tr>
<td>15a.</td>
<td>$(x \cdot y)' = x' + y'$</td>
<td><strong>DeMorgan’s</strong></td>
</tr>
<tr>
<td>15b.</td>
<td>$(x + y)' = x' \cdot y'$</td>
<td><strong>Theorem</strong></td>
</tr>
<tr>
<td>16a.</td>
<td>$x + x' \cdot y = x + y$</td>
<td></td>
</tr>
<tr>
<td>16b.</td>
<td>$x \cdot (x' + y) = x \cdot y$</td>
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</tbody>
</table>
Induction proof of $x + x' \cdot y = x + y$

- Use perfect induction to prove $x + x' \cdot y = x + y$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x'y$</th>
<th>$x + x'y$</th>
<th>$x + y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
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</tr>
</tbody>
</table>

equivalent

Perfect induction example

- Use perfect induction to prove $(xy)' = x' + y'$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$xy$</th>
<th>$(xy)'$</th>
<th>$x'$</th>
<th>$y'$</th>
<th>$x' + y'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
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<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>

equivalent
**Proof (algebraic manipulation)**

- Prove
  - \((X+A)(X'+A)(A+C)(A+D)X = AX\)
  - \((X+A)(X'+A)(A+C)(A+D)X\)
  - \((X+A)(X'+A)(A+CD)X\) (using 12b)
  - \((X+A)(X'+A)(A+CD)X\)
  - \((A)(A+CD)X\) (using 14b)
  - \((A)(A+CD)X\)
  - \(AX\) (using 13b)

**Algebraic manipulation**

- Algebraic manipulation can be used to simplify Boolean expressions
  - Simpler expression => simpler logic circuit
- Not practical to deal with complex expressions in this way
- However, the theorems & properties provide the basis for automating the synthesis of logic circuits in CAD tools
  - To understand the CAD tools the designer should be aware of the fundamental concepts
Venn diagrams

- Venn diagram: graphical illustration of various operations and relations in an algebra of sets
- A set $s$ is a collection of elements that are members of $s$ (for us this would be a collection of Boolean variables and/or constants)
- Elements of the set are represented by the area enclosed by a contour (usually a circle)
Venn diagrams

(e) $XY$

(f) $X+Y$

(g) $XY'$

(h) $XY+Z$

DeMorgan’s Theorem

$(x+y)' = x'y'$

Equivalent Venn diagrams imply equivalent functions
**Notation and terminology**

- Because of the similarity with arithmetic addition and multiplication operations, the ***OR*** and ***AND*** operations are often called the ***logical sum*** and ***product*** operations.

**The expression**
- \( ABC + A'B'D + ACE' \)
- Is a sum of three product terms

**The expression**
- \( (A+B+C)(A'+B+D)(A+C+E') \)
- Is a product of three sum terms

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**Precedence of operations**

- In the absence of parentheses, operations in a logical expression are performed in the order
  - NOT, AND, OR

- Thus in the expression \( AB + A'B' \), the variables in the second term are complemented before being ANDed together. That term is then ORed with the ANDed combination of A and B (the AB term).
Precedence of operations

• Draw the circuit diagrams for the following

  - \( f(a,b,c) = (a'+b)c \)

  - \( f(a,b,c) = a'b+c \)