Introduction to the Fourier transform

- Let $f(x)$ be a continuous function of a real variable $x$
- The Fourier transform of $f(x)$, denoted by $F \{ f(x) \}$ is given by:
  \[
  F \{ f(x) \} = F(u) = \int_{-\infty}^{\infty} f(x) \exp[-j2\pi xu]dx
  \]
- where $j = \sqrt{-1}$
- Given $F(u)$, $f(x)$ can be obtained by using the inverse Fourier transform:
  \[
  F^{-1} \{ F(u) \} = f(x) = \int_{-\infty}^{\infty} F(u) \exp[j2\pi xu]du.
  \]
The Fourier transform (continued)

• These two equations, called the Fourier transform pair, exist if \( f(x) \) is continuous and integrable and \( F(u) \) is integrable.
• These conditions are almost always satisfied in practice.
• We are concerned with functions \( f(x) \) which are real, however the Fourier transform of a real function is, generally, complex. So,

\[
F(u) = R(u) + jI(u)
\]

• where \( R(u) \) and \( I(u) \) denote the real and imaginary components of \( F(u) \) respectively.

The Fourier transform (continued)

• Expressed in exponential form, \( F(u) \) is:

\[
F(u) = |F(u)|e^{j\phi(u)}
\]

• where \( |F(u)| = \sqrt{R^2(u) + I^2(u)} \)

• and \( \phi(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right) \)

• The magnitude function \( |F(u)| \) is called the Fourier spectrum of \( f(x) \)
• and \( \phi(u) \) is the phase angle.
The Fourier transform (continued)

• The square of the spectrum,

\[ P(u) = |F(u)|^2 = R^2(u) + I^2(u) \]

• is commonly called the power spectrum (or the spectral density) of f(x).

• The variable \( u \) is often called the frequency variable. This name arises from the expression of the exponential term \( \exp[-j2\pi ux] \) in terms of sines and cosines (from Euler’s formula):

\[ \exp[-j2\pi ux] = \cos(2\pi ux) - j\sin(2\pi ux) \]

The Fourier transform (continued)

• Interpreting the integral in the Fourier transform equation as a limit summation of discrete terms make it obvious that:
  – \( F(u) \) is composed of an infinite sum of sine and cosine terms.
  – Each value of \( u \) determines the frequency of its corresponding sine-cosine pair.
Fourier transform example

Consider the following simple function. The Fourier transform is:

\[ F(u) = \int_{-\infty}^{\infty} f(x) \exp[-j2\pi ux] dx \]

\[ = \int_{0}^{X} A \exp[-j2\pi ux] dx \]

\[ = \frac{-A}{j2\pi u} [e^{-j2\pi uX}]_0 = \frac{-A}{j2\pi u} [e^{-j2\pi uX} - 1] \]

\[ = \frac{A}{j2\pi u} [e^{j2\pi uX} - e^{-j2\pi uX}] e^{-j\pi X} \]

\[ = \frac{A}{\pi u} \sin(\pi uX) e^{-j\pi X} \]

Fourier transform example (continued)

This is a complex function. The Fourier spectrum is:

\[ |F(u)| = \left| \frac{A}{\pi u} \sin(\pi uX) e^{-j\pi X} \right| \]

\[ = AX \left| \frac{\sin(\pi uX)}{\pi uX} \right| \]

A plot of \(|F(u)|\) looks like the following:
The 2-D Fourier transform

- The Fourier transform can be extended to 2 dimensions:

\[ \mathcal{F}\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp[-j2\pi(ux + vy)] \, dx \, dy. \]

- and the inverse transform

\[ \mathcal{F}^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) \exp[j2\pi(ux + vy)] \, du \, dv. \]

The 2-D Fourier transform (continued)

- The 2-D Fourier spectrum is:

\[ |F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)} \]

- The phase angle is:

\[ \varphi(u,v) = \tan^{-1}\left( \frac{I(u,v)}{R(u,v)} \right) \]

- The power spectrum is:

\[ P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \]
Sample 2-D function and its Fourier spectrum

Example 2-D Fourier transform

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp[-j2\pi(ux + vy)] \, dx \, dy \]

\[ = A \left[ \int_{0}^{\infty} \exp[-j2\pi ux] \, dx \right] \left[ \int_{0}^{\infty} \exp[-j2\pi vy] \, dy \right] \]

\[ = A \left[ \frac{e^{-j2\pi ux}}{-j2\pi u} \right]_{0}^{\infty} \left[ \frac{e^{-j2\pi vy}}{-j2\pi v} \right]_{0}^{\infty} \]

\[ = \frac{A}{-j2\pi} \left[ e^{-j2\pi X} - 1 \right] \frac{1}{-j2\pi} \left[ e^{-j2\pi Y} - 1 \right] \]

\[ = AX[Y \sin(\pi X) e^{-j\pi X}] \frac{\sin(\pi Y)}{(\pi Y)} \]
Example 2-D functions and their spectra

The discrete Fourier transform

- Suppose a continuous function, $f(x)$, is discretized into a sequence
  
  $\{f(x_0), f(x_0 + \Delta x), f(x_0 + 2\Delta x), \ldots, f(x_0 + (N-1)\Delta x)\}$

- by taking $N$ samples $\Delta x$ units apart

- Let $x$ refer to either a continuous or discrete value by saying

  $f'(x) = f'(x_0 + x\Delta x)$

- where $x$ assumes the discrete values 0, 1, ..., $N-1$ and

- $\{f(0), f(1), \ldots, f(N-1)\}$ denotes any $N$ uniformly spaced samples from a corresponding continuous function
The discrete Fourier transform pair

- The discrete Fourier transform is given by:
  \[ F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp[-j2\pi ux / N] \]
- for \( u=0, 1, \ldots ,N-1 \)
- The discrete inverse Fourier transform is given by:
  \[ f(x) = \sum_{u=0}^{N-1} F(u) \exp[j2\pi ux / N] \]
- for \( x=0, 1, \ldots ,N-1 \)
- The values of \( u=0, 1, \ldots ,N-1 \) in the discrete case correspond to samples of the continuous transform at 0, \( \Delta u \), 2\( \Delta u \), \ldots , (N-1)\( \Delta u \)
- \( \Delta u \) and \( \Delta x \) are related by \( \Delta u = 1/(N \Delta x) \)
The 2-D discrete Fourier transform

In the 2-D case:

\[ F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp[-j2\pi(ux/M + vy/N)] \]

for \( u=0 \to M-1 \) and \( v=0 \to N-1 \)

\[ f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \exp[j2\pi(ux/M + vy/N)] \]

for \( x=0 \to M-1 \) and \( y=0 \to N-1 \)

The discrete function \( f(x, y) \) represents samples of the continuous function at \( f(x_0+x\Delta x, y_0+y\Delta y) \)

\[ \Delta u=1/(M\Delta x) \text{ and } \Delta v=1/(N\Delta y) \]

The 2-D discrete Fourier transform (continued)

For the case when \( N=M \) (such as in a square image)

\[ F(u, v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp[-j2\pi(ux+vy)/N] \]

and

\[ f(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) \exp[j2\pi(ux+vy)/N] \]

Note each expression in this case has a 1/N term. The grouping of these constant multiplier terms in the Fourier transform pair is arbitrary.
Discrete Fourier transform example

- Consider sampling at \( x_0 = 0.5, x_1 = 0.75, x_2 = 1.0, \) and \( x_3 = 1.25 \)
- Here \( \Delta x = 0.25 \) and \( x \) ranges from \( 0 \rightarrow 3 \)

Discrete Fourier transform example (continued)

- The four corresponding Fourier transform terms are

\[
\begin{align*}
F(0) &= \frac{1}{4} \sum_{i=0}^{3} f(x) \exp[0] \\
&= \frac{1}{4} [ f(0) + f(1) + f(2) + f(3) ] \\
&= \frac{1}{4} [ 2 + 3 + 4 + 4 ] \\
&= 3.25 \\
F(2) &= -\frac{1}{4} [1 + 0j] \\
F(3) &= -\frac{1}{4} [2 + j] \\

F(0) &= \frac{1}{4} \sum_{i=0}^{3} f(x) \exp[-j2\pi/4] \\
&= \frac{1}{4} [2e^0 + 3e^{j\pi/2} + 4e^{j\pi} + 4e^{j3\pi/2} ] \\
&= \frac{1}{4} [-2 + j] \\
F(1) &= \frac{1}{4} \sum_{i=0}^{3} f(x) \exp[-j\pi/2] \\
&= \frac{1}{4} [2e^0 + 3e^{j\pi/2} + 4e^{j\pi} + 4e^{j3\pi/2} ] \\
&= \frac{1}{4} [2 + j] \\
F(3) &= -\frac{1}{4} [2 + j] \
\end{align*}
\]
Discrete Fourier transform example (continued)

- The Fourier spectrum is then

\[
|F(0)| = 3.25 \\
|F(1)| = [(2/4)^2 + (1/4)^2]^{1/2} = \sqrt{5}/4 \\
|F(2)| = [(1/4)^2 + (0/4)^2]^{1/2} = 1/4 \\
|F(3)| = [(2/4)^2 + (1/4)^2]^{1/2} = \sqrt{5}/4
\]

Properties of the 2-D Fourier transform

- The dynamic range of the Fourier spectra is generally higher than can be displayed
- A common technique is to display the function

\[
D(u,v) = c \log[1 + |F(u,v)|]
\]

where \(c\) is a scaling factor and the logarithm function performs a “compression” of the data
- \(c\) is usually chosen to scale the data into the range of the display device, [0-255] typically ([1-256] for 256 gray-level MATLAB image)
Separability

- The discrete transform pair can be written in separable forms

\[ F(u, v) = \frac{1}{N} \sum_{x=0}^{N-1} \exp\left[-j\frac{2\pi ux}{N}\right] \sum_{y=0}^{N-1} f(x, y) \exp\left[-j\frac{2\pi vy}{N}\right] \]

- for \( u, v = 0, 1, \ldots, N-1 \)

\[ f(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \exp\left[j\frac{2\pi ux}{N}\right] \sum_{v=0}^{N-1} F(u, v) \exp\left[j\frac{2\pi vy}{N}\right] \]

- for \( x, y = 0, 1, \ldots, N-1 \)

- So, \( F(u, v) \) or \( f(x, y) \) can be obtained in 2 steps by successive applications of the 1-D Fourier transform or its inverse.

Separability (continued)

- The 2-D transform can be expressed as

\[ F(u, v) = \frac{1}{N} \sum_{x=0}^{N-1} F(x, v) \exp\left[-j\frac{2\pi ux}{N}\right] \]

- where

\[ F(x, v) = N \left[ \frac{1}{N} \sum_{y=0}^{N-1} f(x, y) \exp\left[-j\frac{2\pi vy}{N}\right] \right] \]

- Graphically, the process is as follows
Translation

The translation properties of the Fourier transform pair are

\[ f(x, y) \exp[j2\pi(u_0x + v_0y)/N] \Leftrightarrow F(u-u_0, v-v_0) \]

and

\[ f(x-x_0, y-y_0) \Leftrightarrow F(u, v) \exp[-j2\pi(ux_0 + vy_0)/N] \]

where the double arrow indicates a correspondence between a function and its Fourier transform (or vice versa).

Multiplying \( f(x, y) \) by the exponential and taking the transform results in a shift of the origin of the frequency plane to the point \((u_0, v_0)\).

Translation (continued)

For our purposes, \( u_0 = v_0 = N/2 \). Therefore,

\[ \exp[j2\pi(u_0x + v_0y)/N] = e^{j\pi(x+y)} \]

\[ = (-1)^{x+y} \]

and

\[ f(x, y)(-1)^{x+y} \Leftrightarrow F(u - N/2, v - N/2) \]

So, the origin of the Fourier transform of \( f(x, y) \) can be moved to the center of the corresponding \( N \times N \) simply by multiplying \( f(x, y) \) by \((-1)^{x+y}\) before taking the transform.

Note: This does not affect the magnitude of the Fourier transform.
Matlab example

%Create data for the test
f=zeros(128);
for x=1:64
    for y=1:64
        f(x,y)=128;
    end
end
% Perform a translation shift on f(x,y)
for x=1:128
    for y=1:128
        f(x,y)=f(x,y)*((-1)^(x+y));
    end
end

Matlab example (continued)

% Compute the 2-D discrete Fourier transform
F=fft2(f);
% Compute the Fourier spectrum
Fspect=sqrt(real(F).^2+imag(F).^2);
% Construct a scaling factor based on
% the dynamic range of the spectrum
FspectMAX=max(max(Fspectrum));
% Compute D, the scaled data
D=(256/(log(1+FspectMAX)))*log(1+Fspect);
figure(1);
% Plot, as an image, a subset of D
image(D(56:74,56:74));colormap(gray(256));
Example image and complete, scaled Fourier spectrum plot

Example image and partial, scaled Fourier spectrum plot (with shifted f(x,y))